

# Higher Topologies in 2+1-Gravity

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## Abstract

I argue that the first-order formalism recently found to describe classical 2+1-Gravity with matter, is also able to include higher topologies. The present gauge, which is conformal with vanishing York time, is characterized by an analytic mapping from single-valued coordinates to Minkowskian ones. In the torus case, this mapping is based on four square-root branch points, whose location is related to the modulus, which has a well defined time dependence. In the general case, it is connected with the hyperelliptic representation of Riemann surfaces.

Classical solutions of (2+1)-Gravity [1] with dynamical matter have recently been found [2, 3] in a (regular) conformal gauge which allows instantaneous propagation of the longitudinal gravitational field. This gauge is characterized by a vanishing York time [4], and by a conformal factor which is of Liouville type, even when arbitrary matter sources are introduced.

Although the pointlike matter described so far is characterized by a non-compact conical geometry at space infinity, corresponding to a total mass of an open universe, in this note I suggest that the method can be extended to some compact cases. Furthermore, I will argue, it may describe proper matter sources as well, possibly combining a non-compact structure with higher topologies.

To set up the present method, I will describe in detail the torus case, that was solved long ago by Moncrief [5] in the (non vanishing) York time gauge. (2+1)-Gravity on a torus is characterized by two nontrivial Poincaré holonomies around the two homotopy cycles, described by the DJH [6] matching conditions

$$X^a)^{II} - B_\alpha^a = \Lambda_b^a(\lambda_\alpha)(X^b)^I - B_\alpha^b, (\alpha = 1, 2). \quad (1)$$

where  $X^a)^I(X^a)^{II}$  denote the (multivalued) Minkowskian coordinates before (after) application of the holonomies. Such Poincaré transformations are constrained to commute, because the compound holonomy around the period parallelogram is trivially contractible, and their Lorentz parts can be chosen to be  $x$ -boosts  $\lambda_1, \lambda_2$ .

The mapping from single-valued coordinates  $x^\mu \equiv (t, z, \bar{z})$  to Minkowskian ones  $X^a \equiv (T, Z, \bar{Z})$  is best expressed in the first-order form

$$dX^a = E_\mu^a(t, z, \bar{z})dx^\mu \quad (2)$$

where the dreibein  $E_\mu^a$ , because of (1), is multivalued and satisfies the monodromy conditions

$$E_\mu^a)^{II} = \Lambda_b^a(\lambda_\alpha)E_\mu^b)^I, (\alpha = 1, 2), \quad (3)$$

along the homotopy cycles, while the metric  $g_{\mu\nu} = E_\mu^a E_{a\nu}$  is not affected by Eq (3) and is thus single-valued.

It is convenient to use an elliptic representation [7] of the torus, and to introduce the  $u$  coordinate, characterized by the holomorphic differential

$$du = \frac{dz}{w(z)}, \quad w^2 = 4z(z-1)(z-a(t)), \quad (4)$$

which provides the customary representation of the torus on the two-sheeted  $z$ -plane, in terms of 4 branch-points at  $z = 0, 1, a(\tau), \infty$ , where

$$\tau = \frac{\omega_1}{\omega_2} = \frac{1}{\sqrt{a}} \frac{F(\frac{1}{2}, \frac{1}{2}; 1; \frac{1}{a})}{F(\frac{1}{2}, \frac{1}{2}; 1; a)} \quad (5)$$

is the modulus and  $2\omega_1, 2\omega_2$  are the torus' periods.

By injecting in Eq (2) the no-torsion and conformal conditions ( $g_{zz} = g_{\bar{z}\bar{z}} = 0$ ) and the Coulomb condition of vanishing York time  $K \sim \partial_z E_z^a + \partial_{\bar{z}} E_{\bar{z}}^a = 0$ , we find that  $E_z^a(E_{\bar{z}}^a)$  is analytic (antianalytic) while  $E_0^a$  is harmonic, with the null-vector representations

$$E_z^a = \frac{N(z, t)}{f'(z, t)} \begin{pmatrix} f \\ 1 \\ f^2 \end{pmatrix}, \quad E_u^a = \frac{n(u, t)}{\dot{f}(u, t)} \begin{pmatrix} f \\ 1 \\ f^2 \end{pmatrix} \quad (6)$$

and the notation

$$N(z(u), t) = \frac{n(u)}{w^2(z)}, \quad \dot{f} = \frac{df}{du} = w(z)f'(z, t). \quad (7)$$

Then the vector monodromies of  $E_u^a$  in Eq (3) are satisfied by imposing projective monodromies on the mapping function  $f$  in Eq (6). Since the  $\lambda$ 's are  $x$ -boosts, we can satisfy Eq (3) by setting

$$f = th\Theta(u), \quad 2\Theta(u + 2\omega_\varrho) = 2\Theta(u) + \lambda_\varrho, \pmod{2\pi ni}, \varrho = 1, 2. \quad (8)$$

Then,  $n(u)$  by Eq (6) and  $\dot{\Theta}(u)$  by Eq (8) are meromorphic functions on the torus, i.e. ellyptic or bi-periodic functions of the  $u$  variable. Notice that  $N(z)$  turns out [2] to be one of the components of the extrinsic curvature, the other one being the (vanishing) York time.

A solution to Eq (8) entails, by Eq (6), a solution to Eqs (2) and (3), which for the combinations  $X_\pm = T \pm X, X_2 = Y$  take the form ( $dt = 0$ )

$$dX_\pm = \pm du \frac{n}{2\dot{\Theta}} e^{\pm 2\Theta} + c.c., \quad dY = du \frac{n}{2i\dot{\Theta}} + c.c., \quad (9)$$

where we can assume  $n$  and  $\dot{\Theta}$  to be even in  $u$ , so that  $Y$  and  $\Theta$  are odd, and  $X_-(u) = X_+(-u)$ . By solving for  $\Theta$  and  $n$ , we can integrate Eq (9) and impose the full DJH matching conditions in Eq (1), including the translational part.

The ellyptic function  $\dot{\Theta}(u)$  is in general given [7] by the ratio of two polynomials in  $z \equiv P(u)$ , the even Weierstrass ellyptic function. The simplest nondegenerate solution is an ellyptic function of order 2, namely

$$2\dot{\Theta} = A + \zeta(u - \alpha) - \zeta(u + \alpha) \sim \tilde{A} \frac{z(u) - \eta}{z(u) - \nu}, \quad (10)$$

where we have introduced the notation

$$\begin{aligned} z(u) &= P(u) = -\dot{\zeta}(u) , \\ \nu &= z(\alpha) = z(-\alpha) \end{aligned} \quad (11)$$

and the  $\zeta$ -function satisfies the monodromies

$$\begin{aligned} \zeta(u + 2\omega_\varrho) &= \zeta(u) + 2\gamma_\varrho , \\ \gamma_1\omega_2 - \gamma_2\omega_1 &= \frac{1}{2}\pi i . \end{aligned} \quad (12)$$

Eq (10) shows simple pole singularities at  $u = \pm\alpha$  ( $z = \nu$ ), which however are harmless, because the integrated  $\Theta$  variable

$$2\Theta(u) = Au + \log \frac{\sigma(u - \alpha)}{\sigma(u + \alpha)} , \quad \left( \zeta = \frac{\dot{\sigma}(u)}{\sigma(u)} \right) , \quad (13)$$

changes by  $\pm 2\pi i$  when turning around  $u = \pm\alpha$ , thus leaving  $\exp(\pm 2\Theta)$  invariant.

Then, by imposing the monodromies in Eq (8) we get the conditions

$$2\omega_\varrho A - 4\gamma_\varrho \alpha = \lambda_\varrho , \quad (\varrho = 1, 2) , \quad (14)$$

which, by exploiting Eq (12), determine  $A$  and  $\alpha$  as functions of  $\lambda_1$  and  $\lambda_2$ :

$$\pi i A = \gamma_1 \lambda_2 - \gamma_2 \lambda_1 , \quad 2\pi i \alpha = \omega_1 \lambda_2 - \omega_2 \lambda_1 . \quad (15)$$

Note that  $\alpha$  vanishes in the degenerate limit  $\tau = \omega_1/\omega_2 \rightarrow \lambda_1/\lambda_2$  (real), in which case the pole at  $z = \nu$  in Eq (10) is avoided, and  $\dot{\Theta}$  reduces to a constant.

In the general case,  $\Theta$  in Eq (13) has a logarithmic singularity at  $u = \pm\alpha$ , while  $e^{\pm 2\Theta}$  shows a pole in the form

$$e^{\pm 2\Theta} = e^{\pm Au} \frac{\sigma(u \mp \alpha)}{\sigma(u \pm \alpha)} . \quad (16)$$

In order to cancel this pole in Eq (9) we shall set

$$n = \frac{1}{2}C(t)\tilde{A}(z(u) - \eta) , \quad \frac{n}{\dot{\Theta}} = C(t)(z(u) - \nu) \quad (17)$$

where  $\eta$  is a common zero of  $n$  and  $\dot{\Theta}$  and is thus [2] an “apparent singularity” [8], and the normalization constant  $C$  will be determined shortly.

Having determined the form of  $n$  and  $\dot{\Theta}$  in Eqs (10) and (17), we can calculate the Schwarzian derivative

$$\{f, z\} = \frac{1}{\omega^2} \left( \{\Theta, u\} - 2\dot{\Theta}^2 \right) + \{u, z\} , \quad (18)$$

which provides the potential of the related Fuchsian problem [8]. The Schwarzian turns out to have 5 singularities, the normal ones at  $z = 0, 1, a(\tau), \infty$  with common difference of exponents  $\mu_\alpha = \frac{m_\alpha}{2\pi} = \frac{1}{2}$ , and the apparent singularity at  $z = \eta$ , which has  $\mu_5 = 2$  and trivial monodromy. The one at  $z = \nu$  is instead absent altogether, and appears to be needed only in the intermediate steps of the construction.

The integration of Eq (9) now proceeds without troubles. Note first that the only singularity of the  $X$ 's comes from the double pole at  $u = 0$ , or  $z = \infty$ , appearing in the expressions

$$\begin{aligned}\frac{dX_+(u)}{du} &= Ce^{Au} \frac{\sigma^2(u - \alpha)}{\sigma^2(\alpha)\sigma^2(u)} + c.c. , \\ \frac{dY(u)}{du} &= \frac{1}{i}C(z(u) - \nu) + c.c. , \quad dX_-(u) = dX_+(-u) .\end{aligned}\tag{19}$$

Expanding around  $u = 0$  we get the behaviour

$$X(u) + iY(u) \simeq -\frac{C}{u} \sim \sqrt{z(u)} , \quad (u \rightarrow 0) ,\tag{20}$$

which provides the same square-root behaviour as for the other branch-points. This checks with the previous result that the “particle masses” at  $z = 0, 1, a, \infty$  are all equal to  $\pi$ .

This remark shows that the handle of the torus is here obtained from quantized pointlike singularities of the extrinsic curvature  $N(z)$  in Eq (7). Furthermore, the present mapping wraps the torus on an infinite 2-dimensional slice of  $X$ -space, rather than on a bounded “cell”.

As a second remark, the equations of motion for  $C(t)$  and  $\tau(t)$  follow from the translational part of the DJH matching conditions in Eq (1). While in the particles' case it is natural to measure time by the clock of one of them, in the present case we just choose to set,  $X_+(-\omega_1 - \omega_2) = t$ , in a somewhat arbitrary way.

Solving for  $Y$  in Eq (19) is simple, and yields

$$Y(u) = -ImC(\zeta(u) + \nu u)\tag{21}$$

so that the translational monodromy in Eq (1) reads

$$Im2C(\gamma_\varrho + \nu\omega_\varrho) = -B_\varrho , \quad (\varrho = 1, 2) .\tag{22}$$

Integrating for  $X_\pm$  is less explicit and yields

$$X_+(u) = t - Re \left( C \int_{-\omega_1 - \omega_2}^u du e^{Au} \frac{\sigma^2(u - \alpha)}{\sigma^2(\alpha)\sigma^2(u)} \right) ,\tag{23}$$

so that Eq (1) provides, after simple algebra, the equations

$$\begin{aligned} & \frac{1}{1 - e^{\lambda_1}} \text{Re} \left( C \int_{-\omega_1 - \omega_2}^{\omega_1 - \omega_2} du e^{Au} \frac{\sigma^2(u - \alpha)}{\sigma^2(\alpha)\sigma^2(u)} \right) = k - t = \\ & = \frac{e^{\lambda_1}}{1 - e^{\lambda_2}} \text{Re} \left( C \int_{\omega_1 - \omega_2}^{\omega_1 + \omega_2} du e^{Au} \frac{\sigma^2(u - \alpha)}{\sigma^2(\alpha)\sigma^2(u)} \right) . \end{aligned} \quad (24)$$

Eqs (22) and (24) provide four real conditions which determine the complex parameters  $C$  and  $\tau$  as functions of time. For instance, in the “quasistatic” limit  $|\lambda_\varrho t| \ll B_\varrho$ ,  $\alpha = 0(\lambda_\varrho)$  in Eq (15) is small, and Eqs (22) and (24) simplify to

$$\text{Im}(2C\nu\omega_\varrho) = -B_\varrho , \quad \text{Re}(2C\nu\omega_\varrho) = \lambda_\varrho(t - k) , \quad (25)$$

thus yielding the expression

$$\tau(t) = \frac{B_1 + i\lambda_1(t - k)}{B_2 + i\lambda_2(t - k)} . \quad (26)$$

which describes a circle, as in Moncrief’s solution [5].

For finite  $\lambda$  values, however, changing from the York time gauge to the Coulomb gauge mixes analytic with antianalytic functions, so that the modulus trajectory is here more complicated than a circle.

Let me now come to the issue of introducing pointlike matter. This can be done either by keeping the compact topology, or by introducing a boundary as well. In the first case, since the holonomy around the period parallelogram is still trivially contractible, we need at least two particles - for instance, static ones with masses  $m$  and  $4\pi - m$ , - so as to yield a trivial compound holonomy. A branch-cut will join the particles and the solution for  $f$  or  $\Theta$  will be found by solving a Fuchsian problem with nontrivial boundary conditions at the edges of the period parallelogram.

The general case of pointlike matter requires a boundary, which without loss of generality can be set around  $u = 0(z = \infty)$  by having all cuts terminating at that point. In this case the trivial contractibility argument relates the holonomy around the boundary to both particle’ momenta and torus’ boosts, much in the same way as it happens in the multiconical geometry [2].

Finally, higher genus topology could be treated following the same method as for the torus. The basic idea is that we could add a handle by adding particles with quantized momenta, - usually not in a unique way - , so as to fit the compact surface requirements. We should therefore get some representation of the higher genus gravitational problem which is much similar to the hyperelliptic ones of Riemann surfaces, with the possible addition of some apparent singularities.

The actual construction of such solutions is still under investigation.

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